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Double-logarithms in Einstein-Hilbert gravity and supergravity

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Abstract: We study the interplay between graviton reggeization and double-logarithmic in energy contributions to four-graviton scattering in theories with and without supersymmetry. Predictions to all orders in the gravitational coupling are given for these double-logarithms. As the number of supersymmetries grows these terms generate a convergent behaviour for the amplitudes at very high energies. At two-loop level, we find agreement with previous exact results for $\mathcal{N} = 8$ supergravity and with those of Boucher-Veronneau and Dixon, who studied the $\mathcal{N} = 4, 5, 6$ supergravities using a conjectured double-copy structure of gravity.

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Contents

1 Introduction	**1**
2 Double-logarithmic approximation	**2**
2.1 One-loop amplitudes	2
2.2 All-loop amplitudes	5
2.2.1 Two-loop truncation and comparison with $N = 4, 5, 6, 8$ results	6
2.2.2 Resummed supergravity amplitudes at high energies	7
2.2.3 Interplay between double logarithms and eikonal contributions	10
3 Conclusions	**13**

1 Introduction

Collider phenomenology and the anti de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] have motivated numerous investigations of scattering amplitudes in gauge and gravitational theories in recent years. In this context, the study of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has been particularly successful since it is possible to investigate loop corrections by only evaluating a reduced set of master topologies [4]. These results for $\mathcal{N} = 4$ SUSY can then be used to also obtain amplitudes in $\mathcal{N} = 8$ supergravity, where the set of minimal topologies is still valid, and to address the question of the renormalizability of the theory at higher orders [5, 6]. This procedure is based on a conjectured double-copy structure of gravity [7] which has recently been applied to $\mathcal{N} = 4, 5, 6$ supergravities at two-loops [8]. In Einstein-Hilbert gravity progress is slower [9–11] since supersymmetry or string theory based techniques [12–14] cannot help.

It is also possible to get important information of the all-orders structure of scattering amplitudes when they are considered in certain kinematical regions. An interesting example is the study of graviton scattering in multi-Regge kinematics (MRK). In this case the amplitudes present a factorized form which can be interpreted in terms of the exchange of reggeized gravitons [15, 16] together with eikonal and double-logarithmic terms [17–19]. These contributions, together with new interaction vertices, can be described by means of a high energy effective action [20]. It is noteworthy that the graviton emission vertex in MRK can be written as a double copy of the corresponding [21–25] QCD gluon emission vertex [17–19, 26].

In the present work double-logarithmic in energy contributions to four-graviton scattering to all orders in the gravitational coupling will be evaluated. This will be done for arbitrary supergravities as well as for Einstein-Hilbert gravity. We will improve previous results based on the resummation of ladder-like diagrams by considering the full set of contributing topologies. The truncation of our results to two loops is in agreement with recent calculations in the literature for $\mathcal{N} = 4, 5, 6, 8$ supergravities. The all-orders resummation of these contributions generates amplitudes which grow with energy when $\mathcal{N} < 4$

and asymptotically tend to zero when $\mathcal{N} > 4$. $\mathcal{N} = 4$ supergravity corresponds to a critical theory where the leading double-logarithmic contributions cancel.

2 Double-logarithmic approximation

For our analysis it is convenient to define the following normalization for the four-point amplitudes

$$\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\mathrm{Born}} \mathcal{M}_{4,(N)}\,, \tag{2.1}$$

$$\mathcal{A}_4^{\mathrm{Born}} = \kappa^2 \frac{s^3}{tu}\,, \tag{2.2}$$

$$\mathcal{M}_{4,(N)} = 1 + \sum_{L=1}^{\infty} \mathcal{M}_{4,(N)}^{(L)}\,, \tag{2.3}$$

where L corresponds to the loop order and N labels the number of gravitinos in the theory. The Mandelstam invariants are $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$ and $u = (p_1-p_4)^2$. $\kappa^2 = 8\pi G$, with G being the Newton's constant.

2.1 One-loop amplitudes

When the one loop amplitude $\mathcal{M}_{4,(N)}^{(1)}$ is calculated in the Regge limit (with $s \gg -t = |q|^2$) the graviton Regge trajectory [17–19]

$$\omega(q) = \frac{\alpha|q|^2}{\pi} \int \frac{d^2k}{|k|^2|q-k|^2} \left(\frac{\left(\vec{k}, \vec{q}-\vec{k}\right)^2}{|k|^2} + \frac{\left(\vec{k}, \vec{q}-\vec{k}\right)^2}{|q-k|^2} - |q|^2 + \frac{N}{2}\left(\vec{k}, \vec{q}-\vec{k}\right) \right) \tag{2.4}$$

appears multiplied by $\ln\left(s/|q|^2\right)$. We have used the notation $\alpha = \kappa^2/(8\pi^2)$. This expression contains both infrared and ultraviolet divergencies which can be regulated by, respectively, the cut-offs λ and Λ, to obtain

$$\omega(q) = -\alpha\,|q|^2 \left(\ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{\Lambda^2}{|q|^2} \right). \tag{2.5}$$

The ultraviolet divergence at $\Lambda \to \infty$ is not fundamental because gravity is renormalizable at one loop. It has a kinematical origin which means that the parameter Λ should be substituted by $\sqrt{s}$, leading to the appearance of the double-logarithmic term $\sim \alpha \ln^2 s$ in the elastic scattering amplitude.

To understand this point in more detail, let us recall that the one loop contribution at high energies can be obtained making use of the Sudakov parametrization for the virtual particle momentum,

$$k = \beta\, p_1 + \alpha\, p_2 + k_\perp\,,\ d^4k = \frac{|s|}{2}\, d\alpha\, d\beta\, d^2k_\perp\,, \tag{2.6}$$

where p_1 and p_2 are the momenta of the colliding particles. Calculating the Feynman integral over α by residues we obtain with leading logarithmic accuracy the following expression

$$\mathcal{A}_4^{(1)}(s,t) \sim \frac{1}{\pi} \int \frac{d^2k_\perp}{|k_\perp|^2} \int_{|q|^2}^{s} \frac{d(\beta s)}{\beta s - |k_\perp|^2}\,. \tag{2.7}$$

It is then clear that the ultraviolet divergence at $|k_\perp| \to \infty$ is absent. In the infrared region of integration $|k_\perp^2| \ll |q|^2$ the above expression factorizes in the form $\mathcal{A}_4^{(1)} \sim \ln\left(|q|^2/\lambda^2\right) \ln\left(s/|q|^2\right)$. This Regge factor containing the infrared divergence can be extracted to all orders in perturbation theory and the amplitude with double-logarithmic accuracy can be conveniently presented using the following Mellin transform:

$$\mathcal{A}_{4,(N)}(s,t) = \mathcal{A}_4^{\text{Born}} \left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2 \ln\frac{|q|^2}{\lambda^2}} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\,\omega}{2\pi i} \left(\frac{s}{|q|^2}\right)^{\omega} \frac{f_\omega^{(N)}}{\omega}\,, \;\delta > 0\,, \tag{2.8}$$

where the tree level amplitude is $\mathcal{A}_4^{\text{Born}} \simeq \kappa^2 s^2/|q|^2$ and the integral over ω contains the contribution of the virtual gluons and gluinos only with $|k_\perp|^2 > |q|^2$. The t-channel partial wave $f_\omega^{(N)}$ in the double-logarithmic approximation can be expanded order by order in perturbation theory,

$$f_\omega^{(N)} = \sum_{n=0}^{\infty} c_n^{(N)} \left(\frac{b}{\omega^2}\right)^n\,, \tag{2.9}$$

where b is the dimensionless parameter

$$b = \alpha |q|^2\,. \tag{2.10}$$

In ref. [17] one of the authors of this work (L.N.L.) made the assumption that double-logarithmic contributions in gravity appear only from ladder diagrams, which allowed him to obtain a closed expression for the scattering amplitude in terms of a Bessel function. However, even in simpler field theories like QED and QCD there is another source of double-logarithmic terms. For example, in e^+e^- backward scattering the diagrams with virtual soft photons emitted and absorbed by the external fermions are essential. These contributions contain, apart from the integration region $|k_\perp|^2 > |q|^2$, the universal infrared divergencies from the region $|k_\perp|^2 \ll |q|^2$ in the form of eq. (2.7). It is then natural to realize that similar non-ladder contributions also play a role in gravity. At one loop they would lead to the following correction to the elastic amplitude

$$\begin{aligned}\frac{\mathcal{A}_{4,\text{soft}}^{(1)}}{\mathcal{A}_4^{\text{Born}}} &= -b \int_{\lambda^2}^{s} \frac{d|k_\perp|^2}{|k_\perp|^2} \int_{|q|^2}^{s} \frac{d(\beta s)}{\beta s - |k_\perp|^2} \\ &= -b \ln\frac{s}{|q|^2}\left(\ln\frac{|q|^2}{\lambda^2} + \frac{1}{2}\ln\frac{s}{|q|^2}\right)\,,\end{aligned} \tag{2.11}$$

which, together with the corresponding ladder correction (including gravitino contributions),

$$\frac{\mathcal{A}_{4,\text{ladder}}^{(1)}}{\mathcal{A}_4^{\text{Born}}} = -b\left(\frac{N-6}{4}\right)\ln^2\frac{s}{|q|^2} \tag{2.12}$$

generate the complete one loop correction (cf. eq. (2.5)):

$$\frac{\mathcal{A}_{4,(N)}^{(1)}}{\mathcal{A}_4^{\text{Born}}} = -b \ln\frac{s}{|q|^2}\left(\ln\frac{|q|^2}{\lambda^2} + \left(\frac{N-4}{4}\right)\ln\frac{s}{|q|^2}\right)\,. \tag{2.13}$$

JHEP07(2014)056

To better understand this result let us now compare it to the exact one-loop amplitudes available in the literature. In ref. [8] Boucher-Veronneau and Dixon used a conjectured double-copy structure of gravity to evaluate four-point scattering amplitudes at two loops in $\mathcal{N} = 4, 5, 6$ supergravities. They also calculated again the well-known $\mathcal{N} = 8$ case.

For $\mathcal{N} = 8$ it is possible to write the exact one-loop amplitude (using $s + t + u = 0$) in the form

$$\begin{aligned}\mathcal{M}^{(1)}_{4,(N=8)} = {} & \underbrace{\alpha\, t \ln\left(\frac{-s}{-t}\right) \ln\left(\frac{-u}{-t}\right)}_{\text{Double Logs}} \\ & \underbrace{+ \alpha\, \frac{t}{2} \ln\left(\frac{-t}{\lambda^2}\right) \left(\ln\left(\frac{-s}{-t}\right) + \ln\left(\frac{-u}{-t}\right)\right)}_{\text{Trajectory}} \\ & \underbrace{- \alpha \frac{(s-u)}{2} \ln\left(\frac{-t}{\lambda^2}\right) \ln\left(\frac{-s}{-u}\right)}_{\text{Eikonal}} . \end{aligned} \tag{2.14}$$

We have indicated the terms which will generate, in the Regge limit ($s \simeq -u$), the double-logarithms, the graviton trajectory and the eikonal pieces.

Following ref. [8] it is possible to relate the previous expression to the $\mathcal{N} = 4, 5, 6$ supergravity amplitudes, i.e.

$$\begin{aligned}\mathcal{M}^{(1)}_{4,(N=4)} = \mathcal{M}^{(1)}_{4,(N=8)} + \alpha\, t \frac{1}{2}\frac{u}{s} \Bigg\{ & \left(2 - \frac{u\,t}{s^2}\right) \left(\ln^2\left(\frac{-u}{-t}\right) + \pi^2\right) \\ & + 1 + \left(\frac{s-u}{s}\right) \ln\left(\frac{-t}{s}\right) + \left(\frac{u-t}{s}\right) \ln\left(\frac{-u}{s}\right) \Bigg\},\end{aligned} \tag{2.15}$$

$$\mathcal{M}^{(1)}_{4,(N=5)} = \mathcal{M}^{(1)}_{4,(N=8)} + \alpha\, t \frac{3}{4}\frac{u}{s} \left(\ln^2\left(\frac{-u}{-t}\right) + \pi^2\right), \tag{2.16}$$

$$\mathcal{M}^{(1)}_{4,(N=6)} = \mathcal{M}^{(1)}_{4,(N=8)} + \alpha\, t \frac{1}{2}\frac{u}{s} \left(\ln^2\left(\frac{-u}{-t}\right) + \pi^2\right). \tag{2.17}$$

From the work of Dunbar and Norridge in ref. [12] we also know the one-loop amplitude in Einstein-Hilbert gravity ($N = 0$):

$$\mathcal{M}^{(1)}_{4,(N=0)} = \mathcal{M}^{(1)}_{4,(N=8)} + \alpha\, t \frac{1}{2}\frac{u}{s} \mathcal{G}\left(\frac{-t}{s}\right), \tag{2.18}$$

with

$$\begin{aligned}\mathcal{G}(x) = {} & \left(4 - 10x + 2x^2 + 15x^3 - 5x^4 - 3x^5 + x^6\right) \\ & \times \left\{ \ln^2(x) + 2x \ln(x) + \pi^2 + \sum_{n=2}^{\infty} x^n \left(\frac{2}{n} \ln(x) + \sum_{l=1}^{n-1} \frac{1}{l(n-l)}\right) \right\} \\ & + \left(\frac{341}{30} - \frac{437}{30}x - \frac{47}{2}x^2 + \frac{37}{3}x^3 + 5x^4 - 2x^5\right) \left(\ln(x) + \sum_{n=1}^{\infty} \frac{x^n}{n}\right) \\ & + \frac{961}{90} + \frac{97}{12}x - \frac{85}{12}x^2 - 2x^3 + x^4 .\end{aligned} \tag{2.19}$$

JHEP07(2014)056

Using these expressions we can see that the double-logarithmic contributions to these amplitudes can be written in the form

$$\mathcal{M}_{4,(N)}^{(1),\mathrm{DL}} = \left(\frac{N-4}{2}\right)\left(\frac{\alpha\, t}{2}\right)\ln^2\left(\frac{s}{-t}\right), \tag{2.20}$$

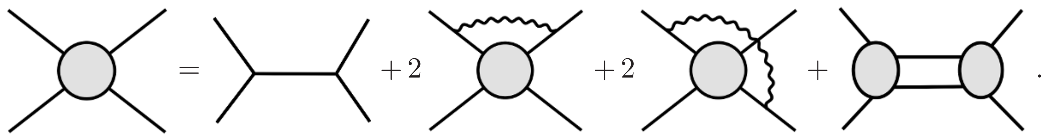
in agreement with our result in eq. (2.13).

2.2 All-loop amplitudes

Since the sources for the double-logarithmic contributions in gravity are the same as in gauge theories (see refs. [27–29] for related calculations in QED and QCD), it is possible to write a similar infrared evolution equation for the partial wave introduced in eq. (2.8), namely,

$$f_\omega^{(N)} = 1 + b\frac{d}{d\,\omega}\frac{f_\omega^{(N)}}{\omega} - b\left(\frac{N-6}{2}\right)\frac{{f_\omega^{(N)}}^2}{\omega^2}, \tag{2.21}$$

which, in pictorial terms, stems from the equation



(2.22)

At the right hand side of eq. (2.21) the first term proportional to b describes the contribution of the virtual graviton with the smallest value of $k_\perp$. The second (ladder) term describes the contribution from the pair with the two softest gravitons or gravitinos exchanged in the t-channel. Let us indicate that, generally, the emission of the virtual soft graviton with transverse momentum $k_\perp$ changes the momentum transfer ($q \to q-k$) for the basic scattering process, and hence modifies the power factors $|q|^{2n} \to |q-k|^{2n}$ of the corresponding amplitude in each order of perturbation theory. Nevertheless, it is correct to neglect the corrections $\sim k_\perp$ because these terms cancel the logarithmic contribution appearing from the integration over $k_\perp$.

It is important to indicate that the coefficients of eq. (2.21) in front of the two terms proportional to b are chosen in such a way as to reproduce the one loop contribution calculated in eq. (2.13). The perturbative solution of the infrared evolution equation (2.21) has the form

$$\begin{aligned} f_\omega^{(N)} = 1 &- \frac{b(N-4)}{2w^2} + \frac{b^2(N-4)(N-3)}{2w^4} \\ &- \frac{b^3(N-4)\left(5N^2-26N+36\right)}{8w^6} \\ &+ \frac{b^4(N-4)\left(7N^3-47N^2+118N-108\right)}{8w^8} \\ &- \frac{b^5(N-4)\left(21N^4-160N^3+556N^2-960N+648\right)}{16w^{10}} + \ldots \end{aligned} \tag{2.23}$$

which leads to the following double-logarithmic asymptotics of the elastic amplitude:

$$\mathcal{A}_{4,(N)}(s,t) = \mathcal{A}_4^{\mathrm{Born}}\left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2\,\ln\frac{|q|^2}{\lambda^2}}\Phi^{(N)}(\xi)\,, \tag{2.24}$$

JHEP07(2014)056

where $\xi = \alpha\,|t|\,\ln^2 \frac{s}{|q|^2}$ and

$$\begin{aligned}
\Phi^{(N)}(\xi) = 1 &- \frac{(N-4)}{2}\frac{\xi}{2} + \frac{(N-4)}{2}(N-3)\frac{\xi^2}{4!} \\
&- \frac{(N-4)}{8}(5N^2-26N+36)\frac{\xi^3}{6!} \\
&+ \frac{(N-4)}{8}\left(7N^3-47N^2+118N-108\right)\frac{\xi^4}{8!} \\
&- \frac{(N-4)}{16}\left(21N^4-160N^3+556N^2-960N+648\right)\frac{\xi^5}{10!} + \dots . \qquad (2.25)
\end{aligned}$$

Before investigating in detail how the amplitudes behave at the light of this all-orders result, let us compare its two-loop truncation with other calculations in the literature.

2.2.1 Two-loop truncation and comparison with $N = 4, 5, 6, 8$ results

In the notation of eq. (2.3), it is well-known that the complete two-loop amplitudes can be written as the sum of two pieces:

$$\mathcal{M}_4^{(2),N} = m_4^{(2),N} + \frac{1}{2}\left(\mathcal{M}_4^{(1),N}\right)^2, \qquad (2.26)$$

where $m_4^{(2),N}$ is infrared finite and has been calculated by Boucher-Veronneau and Dixon in ref. [8] for $N = 4, 5, 6$. The $N = 8$ case was calculated earlier in ref. [12]. We will now present these infrared finite remainders in the Regge limit and with leading double-logarithmic accuracy. First, it is convenient to write the exact amplitudes in the form

$$\begin{aligned}
\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\rm Born}\left(\frac{-t}{\mu^2}\right)^{\alpha t\left(\ln\left(\frac{s}{-t}\right)+i\pi\left(\frac{s}{t}\right)\right)} \\
\times\Bigg\{1 + \left(\frac{N-4}{2}\right)\left(\frac{\alpha\, t}{2}\right)\ln^2\left(\frac{s}{-t}\right) \\
+\frac{1}{2}\left(\frac{N-4}{2}\right)^2\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right) + m_{4,\rm DL}^{(2),N} + \dots\Bigg\}, \qquad (2.27)
\end{aligned}$$

where we have exponentiated the infrared divergent terms and singled out the non-exponentiating double-logarithmic contributions. The latter contain those pieces related to the one-loop result and its square, following the last term of eq. (2.26), and the $m_{4,\rm DL}^{(2),N}$ contribution, which can be read off ref. [8]:

$$m_{4,\rm DL}^{(2),N=4} = 0\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right) \Longrightarrow 0, \qquad (2.28)$$

$$m_{4,\rm DL}^{(2),N=5} = \frac{1}{24}\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right) \Longrightarrow \frac{1}{6}\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right), \qquad (2.29)$$

$$m_{4,\rm DL}^{(2),N=6} = 0\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right) \Longrightarrow \frac{1}{2}\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right), \qquad (2.30)$$

$$m_{4,\rm DL}^{(2),N=8} = -\frac{1}{3}\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right) \Longrightarrow \frac{5}{3}\left(\frac{\alpha\, t}{2}\right)^2\ln^4\left(\frac{s}{-t}\right). \qquad (2.31)$$

At the right hand side of these expressions we have written the final double-logarithmic contribution to the amplitude. It is important to note that these results are in complete agreement with the two-loop truncation of our prediction for any N in eq. (2.25). Higher order terms for different supergravities or Einstein-Hilbert gravity can be obtained from it. This should serve as a useful test of multi-loop calculations of four-graviton amplitudes. As an example, in $\mathcal{N}=8$ SUGRA we obtain

$$\begin{aligned} \mathcal{A}_{4,(\mathcal{N}=8)} = \mathcal{A}_4^{\rm Born} & \left(\frac{-t}{\mu^2}\right)^{\alpha t \left(\ln\left(\frac{s}{-t}\right) + i\pi\left(\frac{s}{t}\right)\right)} \\ & \times \left\{ 1 + 2\left(\frac{\alpha\, t}{2}\right)\ln^2\left(\frac{s}{-t}\right) + \frac{5}{3}\left(\frac{\alpha\, t}{2}\right)^2 \ln^4\left(\frac{s}{-t}\right) \right. \\ & \quad + \frac{37}{45}\left(\frac{\alpha\, t}{2}\right)^3 \ln^6\left(\frac{s}{-t}\right) + \frac{353}{1260}\left(\frac{\alpha\, t}{2}\right)^4 \ln^8\left(\frac{s}{-t}\right) \\ & \quad \left. + \frac{583}{8100}\left(\frac{\alpha\, t}{2}\right)^5 \ln^{10}\left(\frac{s}{-t}\right) + \dots \right\}. \end{aligned} \tag{2.32}$$

Now we turn to study the high energy asymptotic behaviour of the resummed amplitudes.

2.2.2 Resummed supergravity amplitudes at high energies

In terms of double-logarithmic contributions the simplest amplitude is that of $\mathcal{N}=4$ SUGRA since their contribution adds to zero and we have the pure Regge asymptotic behaviour

$$\mathcal{A}_{4,(\mathcal{N}=4)}(s,t) = \mathcal{A}_4^{\rm Born}\left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2\,\ln\frac{|q|^2}{\lambda^2}}. \tag{2.33}$$

In the case of $\mathcal{N}=6$ SUGRA eq. (2.21) can be solved in the form

$$f_\omega^{(N=6)} = \int_0^\infty d\,z\, e^{-z}\, e^{-\frac{z^2 b}{2\,\omega^2}}, \tag{2.34}$$

obtaining the following result for the amplitude

$$\mathcal{A}_{4,(\mathcal{N}=6)}(s,t) = \mathcal{A}_4^{\rm Born}\left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2\,\ln\frac{|q|^2}{\lambda^2}} \exp\left(-\frac{\alpha\,|q|^2}{2}\ln^2\frac{s}{|q|^2}\right). \tag{2.35}$$

In the general case with arbitrary N it is useful to introduce the new function $y(x)$ and the new variable x according to the definitions

$$f_\omega^{(N)} = \frac{2x}{6-N}\, y^{(N)}(x)\,, \quad x = \frac{\omega}{\sqrt{b}}\,, \tag{2.36}$$

to reduce our eq. (2.21) to the Riccati equation

$$y^{(N)\prime}(x) + y^{(N)2}(x) - x\, y^{(N)} + \frac{6-N}{2} = 0\,. \tag{2.37}$$

By introducing the new function $\Psi_{(N)}(x)$ as follows

$$y^{(N)} = \frac{d}{d\,x}\ln\left(e^{\frac{x^2}{4}}\,\Psi_{(N)}(x)\right) \tag{2.38}$$

JHEP07(2014)056

we obtain for it a linear Schrödinger equation, i.e.

$$\left(-\frac{d^2}{dx^2}+\frac{N-7}{2}+\frac{x^2}{4}\right)\Psi_{(N)}(x)=0\,. \tag{2.39}$$

For even values of N we have the following simple solutions for this equation:

$$\Psi^{(1)}_{(N=8)}(x)=e^{\frac{x^2}{4}}\,, \tag{2.40}$$

$$\Psi^{(1)}_{(N=6)}(x)=e^{-\frac{x^2}{4}}\,, \tag{2.41}$$

$$\Psi^{(1)}_{(N=4)}(x)=x\,e^{-\frac{x^2}{4}}\,, \tag{2.42}$$

$$\Psi^{(1)}_{(N=2)}(x)=\left(1-x^2\right)e^{-\frac{x^2}{4}}\,, \tag{2.43}$$

$$\Psi^{(1)}_{(N=0)}(x)=x\left(3-x^2\right)e^{-\frac{x^2}{4}}\,. \tag{2.44}$$

The above solutions for $N=4,2,0$ are physical and therefore it is possible to calculate the following partial wave $f^{(N)}_\omega$ for these cases

$$\frac{f^{(N=4)}_\omega}{\omega}=\frac{1}{\omega}\,, \tag{2.45}$$

$$\frac{f^{(N=2)}_\omega}{\omega}=\frac{1/2}{\omega+\sqrt{b}}+\frac{1/2}{\omega-\sqrt{b}}\,, \tag{2.46}$$

$$\frac{f^{(N=0)}_\omega}{\omega}=\frac{1/3}{\omega}+\frac{1/3}{\omega+\sqrt{3b}}+\frac{1/3}{\omega-\sqrt{3b}}\,. \tag{2.47}$$

These poles in the ω-plane lead to the following double-logarithmic asymptotic behavior of the corresponding scattering amplitudes:

$$\mathcal{A}_{4,(N)}(s,t)=\mathcal{A}_4^{\text{Born}}\left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2\,\ln\frac{|q|^2}{\lambda^2}}\,r^{(N)}(s,t), \tag{2.48}$$

$$r^{(N)}(s,t)=\int_{\delta-i\infty}^{\delta+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{-t}\right)^{\omega}\frac{f^{(N)}_\omega}{\omega}, \tag{2.49}$$

where

$$r^{(N=4)}(s,t)=1\,, \tag{2.50}$$

$$r^{(N=2)}(s,t)=\frac{1}{2}\left(\left(\frac{s}{|q|^2}\right)^{\sqrt{\alpha|q|^2}}+\left(\frac{s}{|q|^2}\right)^{-\sqrt{\alpha|q|^2}}\right)\,, \tag{2.51}$$

$$r^{(N=0)}(s,t)=\frac{1}{3}\left(1+\left(\frac{s}{|q|^2}\right)^{\sqrt{3\alpha|q|^2}}+\left(\frac{s}{|q|^2}\right)^{-\sqrt{3\alpha|q|^2}}\right)\,. \tag{2.52}$$

The second solution of the Schrödinger equation for even N can be constructed with the use of the Wronskian for two independent solutions

$$\Psi^{(1)}_{(N)}\frac{d}{dx}\Psi^{(2)}_{(N)}-\Psi^{(2)}_{(N)}\frac{d}{dx}\Psi^{(1)}_{(N)}=\text{constant}\,. \tag{2.53}$$

By integrating it we obtain the physical solution for $\mathcal{N}=8$ supergravity:

$$\Psi_{(N=8)}(x) = e^{\frac{x^2}{4}} \int_x^\infty dz\, e^{-\frac{z^2}{2}} = e^{-\frac{x^2}{4}} \int_0^\infty dy\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\,, \tag{2.54}$$

which is proportional to the probability integral $\mathrm{erfc}(\sqrt{2}x)$. We can then obtain the corresponding scattering amplitude in the form

$$r^{(8)}(s,t) = -\int_{a-i\infty}^{a+i\infty} \frac{d\,x}{2\pi i} \left(\frac{s}{|q|^2}\right)^{x\sqrt{b}} \frac{d}{d\,x} \ln \int_0^\infty dy\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\,. \tag{2.55}$$

Generally, the physical solution of the Schrödinger eq. (2.39) can be expressed in terms of the parabolic cylinder function

$$\Psi_{(N)}(x) = D_{\frac{6-N}{2}}(x)\,,\;\; D_\nu(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\nu)} \int_0^\infty \frac{dy}{y^{\nu+1}}\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\,. \tag{2.56}$$

Therefore

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N} \frac{1}{\sqrt{b}} \frac{d}{d\,x} \ln\left(\int_0^\infty \frac{dy}{y^{\frac{8-N}{2}}}\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\right)\,, \quad x = \frac{\omega}{\sqrt{b}}\,. \tag{2.57}$$

The integral over y is convergent for $\nu < 0$. For example, for $N=7$ we have

$$\frac{f_\omega^{(7)}}{\omega} = -\frac{2}{\sqrt{b}} \frac{d}{d\,x} \ln\left(\int_0^\infty \frac{dy}{\sqrt{y}}\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\right)\,, \quad x = \frac{\omega}{\sqrt{b}}\,. \tag{2.58}$$

In the general case with arbitrary ν we can choose the integration contour L in the complex plane y which goes from $y=+\infty$, surrounds the singular point $y=0$ and returns again to $y=+\infty$:

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N} \frac{1}{\sqrt{b}} \frac{d}{d\,x} \ln\left(\frac{1}{2\pi i} \int_L \frac{dy}{(-y)^{\frac{8-N}{2}}}\, e^{-\frac{y^2}{2}}\, e^{-x\,y}\right)\,, \quad x = \frac{\omega}{\sqrt{b}}\,. \tag{2.59}$$

In particular, for $N \to 6$ we obtain

$$\frac{f_\omega^{(6)}}{\omega} = \frac{1}{\sqrt{b}} \int_0^\infty dy\, e^{-\frac{y^2}{2}}\, e^{-x\,y} = \frac{1}{\omega} \int_0^\infty d\,z\, e^{-z}\, e^{-\frac{z^2 b}{2\,\omega^2}}\,, \tag{2.60}$$

which is in agreement with eq. (2.34). After differentiating over x in eq. (2.59) and taking $N = 4, 2, 0$ we can also reproduce our previous results in eqs. (2.45), (2.46), (2.47).

For odd values of N and $N=8$ the function $\Psi_{(N)}(x)$ has an infinite number of zeros situated asymptotically close to the lines $\arg z = \pm\frac{3}{4}\pi$. The trajectories of these Regge poles satisfy the following equation at large n

$$x^{(N)\,2} \approx -2(7-N)\,\ln x^{(N)} + 2\,\pi\, e^{\pm\frac{3}{2}\pi i}\,(2n+1)\,,\;\; n = 0, 1, 2, \ldots\,. \tag{2.61}$$

For $\mathcal{N}=1$ SUGRA there are three Regge poles with the real trajectories

$$x_1^{(N=1)} \approx -2.460\,, \quad x_2^{(N=1)} \approx -0.452\,, \quad x_3^{(N=1)} = 1.402\,, \tag{2.62}$$

JHEP07(2014)056

leading to the following growth of the scattering amplitude:

$$\lim_{s\to\infty} r^{(N=1)}(s,t) \approx \frac{2}{5}\left(\frac{s}{|q|^2}\right)^{1.402\sqrt{\alpha}\,|q|} . \tag{2.63}$$

In $\mathcal{N}=3$ SUGRA there are two real Regge trajectories,

$$x_1^{(N=3)} \approx -1.747\,, \qquad x_2^{(N=3)} \approx 0.5508\,, \tag{2.64}$$

also generating a growing contribution to the amplitude:

$$\lim_{s\to\infty} r^{(N=3)}(s,t) \approx \frac{2}{3}\left(\frac{s}{|q|^2}\right)^{0.5508\sqrt{\alpha}\,|q|} . \tag{2.65}$$

For the case $N=5$ there only exists one Regge pole with the real Regge trajectory

$$x_1^{(N=5)} \approx -0.762\,, \tag{2.66}$$

which leads to an amplitude falling with energy (together with some oscilating contributions from the poles in the complex plane)

$$\lim_{s\to\infty} r^{(N=5)}(s,t) \approx 2\left(\frac{s}{|q|^2}\right)^{-0.762\sqrt{\alpha}\,|q|} . \tag{2.67}$$

In the cases $N=7,8$ all the poles of f_ω are situated in the left hand side of the complex ω-plane and therefore $r^{(N)}(s,t)$ here tends to zero when $s\to\infty$. In particular,

$$\lim_{s\to\infty} r^{(N=8)}(s,t) = 2\left(\frac{s}{|q|^2}\right)^{-1.916\sqrt{\alpha}|q|} \cos\left(2.8164\sqrt{\alpha}|q|\ln\frac{s}{|q|^2}\right) . \tag{2.68}$$

Besides these asymptotic estimates, we have performed an exact numerical analysis of the function $r^{(N)}(s,t)$ (see eq. (2.49)). For $N=0,1,2,3,4,5,6,7,8$, in agreement with the previous analysis in this section, we show the energy behaviour of the scattering amplitudes for different supergravities in figure 1. The monotonically growing with energy solutions for $N<4$ are shown in more detail in figure 2. The critical solution at $N=4$ is flat with energy. The two monotonically decreasing with energy solutions for $N=5,6$ are given in figure 3 and the two oscillatory and decreasing with energy ones for $N=7,8$ can be seen in figure 4.

It is interesting to point out that the falling asymptotic behavior of the amplitudes at $s\to\infty$ in the $\mathcal{N}=5,6,7,8$ supergravities is likely to be related to good ultraviolet properties of these theories, including the possible renormalizability of $\mathcal{N}=8$ SUGRA.

2.2.3 Interplay between double logarithms and eikonal contributions

In this section we study the interplay between the double logarithmic terms just resummed and the eikonal factor. In particular, we are looking for the coefficient of the term $\sim \alpha^2 s\, t \ln^3\left(\frac{s}{-t}\right)$ in the two-loop amplitude of eq. (2.26) (with the Born term factorized), which corresponds to the leading piece proportional to s. In $\mathcal{M}_4^{(2),N}$ there is another term of the form $\sim \alpha^2 s^2 \ln^3\left(\frac{-t}{\lambda^2}\right)$ which stems from the simple exponentiation of the one-loop amplitude.

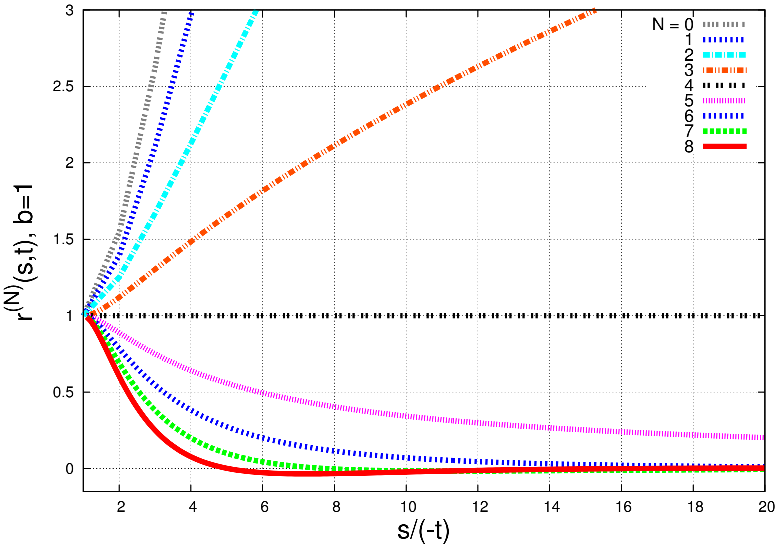


Figure 1. Scattering amplitude for $N = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

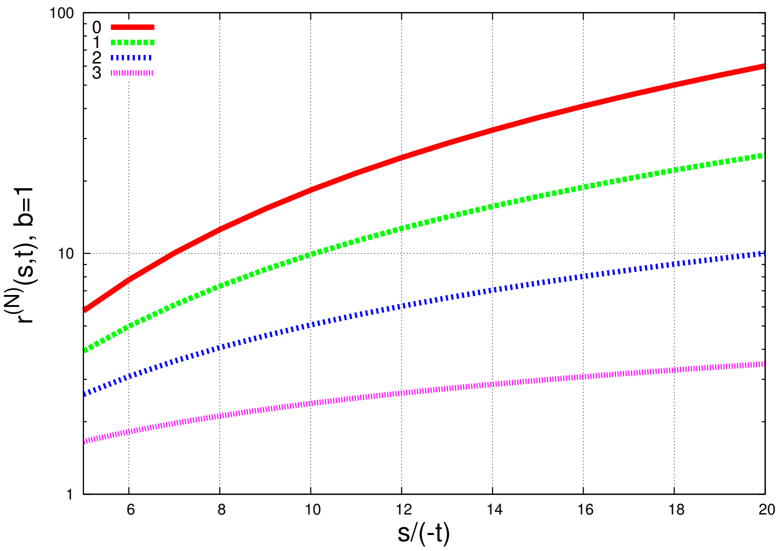


Figure 2. Scattering amplitude for $N = 0, 1, 2, 3$.

We can calculate these contributions writing the eikonal representation for the four point amplitude with the exponentiated Regge-like infrared divergent factor, i.e.,

$$A_{\text{eik,DL}}(s,t) = -2is \left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2 \ \ln \frac{|q|^2}{\lambda^2}} \int d^2\vec{\rho}\, e^{i\vec{q}\cdot\vec{\rho}} \left(e^{i\delta(\vec{\rho},\ln s)} - 1\right), \tag{2.69}$$

where the eikonal phase $\delta\left(\vec{\rho}, \ln s\right)$, with double logarithmic accuracy, is given by

$$\delta\left(\vec{\rho}, \ln s\right) = \frac{s}{2}\frac{\kappa^2}{(2\pi)^2} \int \frac{d^2\vec{q}}{|\vec{q}|^2} e^{-i\vec{q}\cdot\vec{\rho}} \Phi\left(\xi\right), \tag{2.70}$$

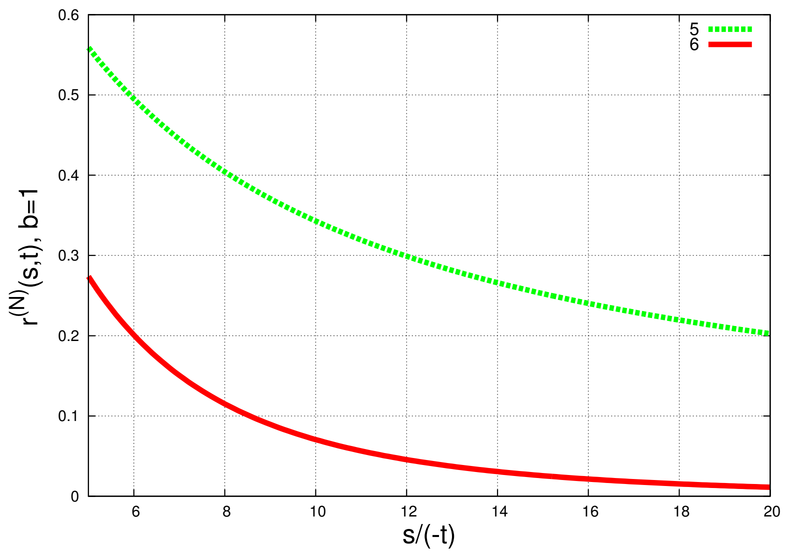


Figure 3. Scattering amplitude for $N = 5, 6$.

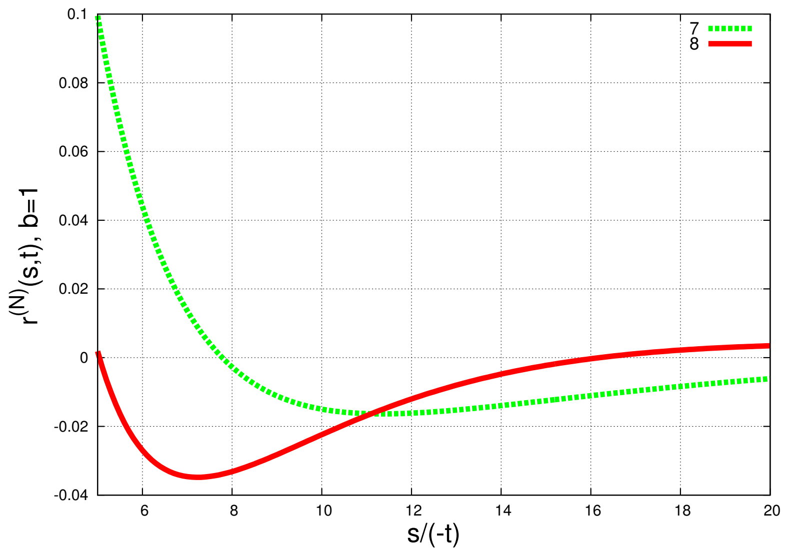


Figure 4. Scattering amplitude for $N = 7, 8$.

with the perturbative expansion of $\Phi(\xi)$, for arbitrary N, written as in eq. (2.25). Within double logarithmic accuracy, we can rewrite $A_{\rm eik,DL}(s,t)$ in the convenient form

$$A_{\rm eik,DL}(s,t) = -2is\left(\frac{s}{|q|^2}\right)^{-\alpha|q|^2\,\ln\frac{|q|^2}{\lambda^2}}(2\pi)^2 \times\sum_{n=0}^{\infty}\frac{(i\alpha s)^n}{n!}\int\prod_{r=1}^{n}\frac{d^2\vec{q}_r}{|\vec{q}_r|^2}\Phi(\xi_r)\,\theta\left(s-|\vec{q}_r|^2\right)\delta\left(\vec{q}-\sum_{l=1}^{n}\vec{q}_l\right). \tag{2.71}$$

This all-orders prediction for the interplay between the eikonal factor and the double logarithms in energy is in agreement with the two-loop result obtained using the color-kinematics duality of ref. [8]:

$$A^{(2)}_{\mathrm{eik,DL}}(s,t) = \kappa^2 \left(\frac{s}{t}\right)^2 (-i\pi s)\alpha^2 t^2 \frac{(N-4)}{12} \ln^3\left(\frac{s}{-t}\right). \tag{2.72}$$

The eikonal phase in eq. (2.70) can be related to the problem of black hole formation and the construction of unitary amplitudes in ultrahigh-energy scattering (see [30, 31] and references therein), a connection which we will discus in a future work.

3 Conclusions

We have calculated the leading double-logarithmic in energy contributions to four-graviton scattering to all orders in the gravitational coupling. These terms are subleading with respect to eikonal contributions but important to understand the high energy behaviour of the scattering amplitudes. Our results are valid for any supergravity as well as for Einstein-Hilbert gravity. We have used infrared evolution equations which take into account both ladder and non-ladder topologies. The truncation of our resummation to two loops is in exact agreement with recent calculations in the literature for $\mathcal{N} = 4, 5, 6, 8$ supergravities. Our results show a growth with energy for the amplitudes when $\mathcal{N} < 4$, a critical invariance with the energy for $\mathcal{N} = 4$, and an asymptotic approach to zero when $\mathcal{N} > 4$.

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References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [INSPIRE].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [INSPIRE].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[4] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The Complete Four-Loop Four-Point Amplitude in $N = 4$ super-Yang-Mills Theory*, *Phys. Rev.* **D 82** (2010) 125040 [arXiv:1008.3327] [INSPIRE].

[5] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys.* **B 530** (1998) 401 [hep-th/9802162] [INSPIRE].

[6] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The Ultraviolet Behavior of $N = 8$ Supergravity at Four Loops*, *Phys. Rev. Lett.* **103** (2009) 081301 [arXiv:0905.2326] [INSPIRE].

[7] Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, *Living Rev. Rel.* **5** (2002) 5 [gr-qc/0206071] [INSPIRE].

[8] C. Boucher-Veronneau and L.J. Dixon, *$N \geq 4$ Supergravity Amplitudes from Gauge Theory at Two Loops*, *JHEP* **12** (2011) 046 [arXiv:1110.1132] [INSPIRE].

[9] J.F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, *Phys. Rev.* **D 50** (1994) 3874 [gr-qc/9405057] [INSPIRE].

[10] N.E. Bjerrum-Bohr, J.F. Donoghue and B.R. Holstein, *Quantum gravitational corrections to the nonrelativistic scattering potential of two masses*, *Phys. Rev.* **D 67** (2003) 084033 [*Erratum ibid.* **D 71** (2005) 069903] [hep-th/0211072] [INSPIRE].

[11] J.F. Donoghue and T. Torma, *Infrared behavior of graviton-graviton scattering*, *Phys. Rev.* **D 60** (1999) 024003 [hep-th/9901156] [INSPIRE].

[12] D.C. Dunbar and P.S. Norridge, *Calculation of graviton scattering amplitudes using string based methods*, *Nucl. Phys.* **B 433** (1995) 181 [hep-th/9408014] [INSPIRE].

[13] Z. Bern, D.C. Dunbar and T. Shimada, *String based methods in perturbative gravity*, *Phys. Lett.* **B 312** (1993) 277 [hep-th/9307001] [INSPIRE].

[14] Z. Bern and D.C. Dunbar, *A mapping between Feynman and string motivated one loop rules in gauge theories*, *Nucl. Phys.* **B 379** (1992) 562 [INSPIRE].

[15] M.T. Grisaru, P. van Nieuwenhuizen and C.C. Wu, *Reggeization and the Question of Higher Loop Renormalizability of Gravitation*, *Phys. Rev.* **D 12** (1975) 1563 [INSPIRE].

[16] M.T. Grisaru and H.J. Schnitzer, *Dynamical Calculation of Bound State Supermultiplets in $N = 8$ Supergravity*, *Phys. Lett.* **B 107** (1981) 196 [INSPIRE].

[17] L.N. Lipatov, *Graviton reggeization*, *Phys. Lett.* **B 116** (1982) 411 [INSPIRE].

[18] L.N. Lipatov, *Multi-Regge Processes in Gravitation*, *Sov. Phys. JETP* **55** (1982) 582 [INSPIRE].

[19] L.N. Lipatov, *High-energy scattering in QCD and in quantum gravity and two-dimensional field theories*, *Nucl. Phys.* **B 365** (1991) 614 [INSPIRE].

[20] L.N. Lipatov, *Effective action for the Regge processes in gravity*, *Phys. Part. Nucl.* **44** (2013) 391 [arXiv:1105.3127] [INSPIRE].

[21] L.N. Lipatov, *Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories*, *Sov. J. Nucl. Phys.* **23** (1976) 338 [INSPIRE].

[22] V.S. Fadin, E.A. Kuraev and L.N. Lipatov, *On the Pomeranchuk Singularity in Asymptotically Free Theories*, *Phys. Lett.* **B 60** (1975) 50 [INSPIRE].

[23] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Multi-Reggeon Processes in the Yang-Mills Theory*, *Sov. Phys. JETP* **44** (1976) 443 [INSPIRE].

JHEP07(2014)056

[24] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories*, *Sov. Phys. JETP* **45** (1977) 199 [INSPIRE].

[25] I.I. Balitsky and L.N. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, *Sov. J. Nucl. Phys.* **28** (1978) 822 [INSPIRE].

[26] A. Sabio Vera, E. Serna Campillo and M.A. Vazquez-Mozo, *Graviton emission in Einstein-Hilbert gravity*, *JHEP* **03** (2012) 005 [arXiv:1112.4494] [INSPIRE].

[27] R. Kirschner and L.N. Lipatov, *Double Logarithmic Asymptotics of Quark Scattering Amplitudes With Flavor Exchange*, *Phys. Rev.* **D 26** (1982) 1202 [INSPIRE].

[28] R. Kirschner and L. Lipatov, *Doubly logarithmic asymptotic of the quark scattering amplitude with nonvacuum exchange in the T channel*, *Sov. Phys. JETP* **56** (1982) 266 [INSPIRE].

[29] R. Kirschner and L. Lipatov, *Double Logarithmic Asymptotics and Regge Singularities of Quark Amplitudes with Flavor Exchange*, *Nucl. Phys.* **B 213** (1983) 122 [INSPIRE].

[30] S.B. Giddings, *The gravitational S-matrix: Erice lectures*, arXiv:1105.2036 [INSPIRE].

[31] M. Ciafaloni and D. Colferai, *Quantum Tunneling and Unitarity Features of an S-matrix for Gravitational Collapse*, *JHEP* **12** (2009) 062 [arXiv:0909.4523] [INSPIRE].

JHEP07(2014)056